

Topology of the Universe

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Abstract

General relativity is unable to determine the topology of the Universe. We propose to apply quantum approach. Quantization of dynamics of a test particle is sensitive to the spacetime topology. Presented results for a particle in de Sitter spacetimes favor a finite universe.

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Specification of the topology of the Universe is necessary for the correct interpretation of cosmological observational data. The problem is that in case our space is multiply connected or/and compact one can see multiple images of cosmic objects by looking along different null geodesics. The topology is also crucial for the description of the birth of the Universe. The resolution of the problem of Universe's large-scale homogeneity may depend on the specification of the topology as well. Finally, determination of the topology would answer the old question of a finite or infinite Universe[1,2].

The Einstein equations being partial differential equations cannot specify the topology of spacetime but only its local properties. In fact, there are always many topologically distinct universe models consistent with a given local geometry. The determination of the topology requires some understanding beyond general relativity.

We propose to apply quantum approach. It turns out that imposition of a quantum formalism upon the classical dynamics of a test particle is sensitive to the choice of the spacetime topology [3,4,5]. In what follows we shall demonstrate this dependence for very simple cases of a test particle in two de Sitter spacetimes having the same local geometries but completely different topologies. The two considered spacetimes are $V_p := (R^1 \times R^1, g)$

and $V_h := (R^1 \times S^1, g)$. In both cases the metric g is defined by the line-element

$$ds^2 = dt^2 - e^{2mt} dx^2, \quad (1)$$

where $m^2 = \Lambda/3$ and $\Lambda > 0$ is a cosmological constant.

In case of V_p the space has the topology R^1 , so it is non-compact and simply connected. The V_h is defined to be a one-sheet hyperboloid embedded in 3d Minkowski space. Since the space of V_h has the topology S^1 , it is compact and multiply connected. There exists an isometric immersion of V_p into V_h [6]. It is defined by the mapping

$$V_p \ni (t, x) \longrightarrow (y^0, y^1, y^2) \in V_h, \quad (2)$$

where

$$\begin{aligned} 2my^0 &:= e^{mt} - e^{-mt} + (mx)^2 e^{mt}, \\ 2my^1 &:= e^{mt} + e^{-mt} - (mx)^2 e^{mt}, \\ y^2 &:= xe^{mt} \end{aligned}$$

and one has

$$(y^2)^2 + (y^1)^2 - (y^0)^2 = (1/m)^2.$$

Action integral describing a relativistic test particle of mass m_0 in gravitational field is defined by

$$S := -m_0 \int d\tau \sqrt{g_{\mu\nu}(x^0(\tau), x^1(\tau)) \dot{x}^\mu(\tau) \dot{x}^\nu(\tau)}, \quad (3)$$

where τ is an evolution parameter, $(x^0, x^1) = (t, x)$ are spacetime coordinates, $\dot{x}^\mu := dx^\mu/d\tau$ ($\mu = 0, 1$), and $g_{\mu\nu}$ ($\mu, \nu = 0, 1$) are metric tensor components. It is known that Hamilton's principle applied to (3) leads to geodesic equations. The set of all solutions to the geodesic equations defines an extended phase-space of the system. The physical phase-space is its subset [4]. It consists of all time-like geodesics which are admitted by the spacetime topology and which satisfy a gauge condition resulting from reparametrization invariance of the action integral (3).

Let us find a general form of a geodesic curve of our test particle. Instead of solving the geodesic equations directly we use local symmetry of the system and apply Noether's theorem. Obtained expressions for dynamical integrals depend explicitly on particle spacetime coordinates and can be converted into expressions for spacetime coordinates in terms of dynamical integrals.

This way one finds a general form of geodesic curve of a test particle. In both cases, V_p and V_h , infinitesimal symmetry transformations lead to three dynamical integrals satisfying $sl(2, R)$ algebra [4,5]. Since the three integrals are constants of motion, it is natural to choose them to represent classical observables of the system. The integrals are functionally dependent since the action (3) is reparametrization invariant. This gauge condition constrains the observables to a one-sheet hyperboloid (OSH). Each point of OSH defines a particle trajectory. However, the sets of all possible trajectories (the physical phase-space) of the two considered systems are quite different. The topology of V_h admits trajectories defined by any point of OSH. Thus the symmetry group of the system is $SO_{\uparrow}(2, 1)$ [5]. In contrary to the V_h case, topology of V_p restricts particle trajectories to the plane. As a result, one infinite curve of points on OSH is not available for particle dynamics and the physical phase-space is now isomorphic to R^2 . Consequently, some global transformations generated by $sl(2, R)$ algebra are not well defined and $SO_{\uparrow}(2, 1)$ is no longer the symmetry group of the system (only translations and dilatations are the symmetry transformations) [4].

Difference between V_p and V_h systems shown at the classical level lead to dramatic difference at the quantum level. Quantization in considered cases means finding representation of $sl(2, R)$ algebra of observables on a Hilbert space having at least two properties: (i) it is a self-adjoint representation and (ii) it is a unitary representation of the symmetry group of the corresponding classical system.

In the case of V_h system there exists the representation of $sl(2, R)$ algebra on the Hilbert space $L^2(S^1)$ which can be lifted to the unitary representation of $SO_{\uparrow}(2, 1)$ group [5]. The V_p case is quite different. The representation of $sl(2, R)$ algebra is parametrized by continuous parameter $\alpha \in S^1$. Each choice of α defines a unitary representation of the universal covering group $\widetilde{SL}(2, R)$ on the Hilbert space $L^2([0, 2\pi])$ and representations corresponding to different α 's are unitarily nonequivalent [4].

In summary, the two considered systems are locally the same but they differ globally due to different topologies of corresponding spacetimes. In case spacetime has topology $R^1 \times S^1$ there is one-to-one correspondence between classical and quantum levels. The canonical quantization applied to the system having spacetime with topology $R^1 \times R^1$ leads to infinitely many unitarily nonequivalent quantum systems. Therefore, presented results for the two 'toy models' favor a finite universe.

It seems that our results can be generalized to the cases of a particle

dynamics in 4d de Sitter spacetimes with $R^1 \times R^3$ and $R^1 \times S^3$ topologies, since the formulae for the line element (1) and the isometric immersion map (2) extend to 4d only by the increase of the number of space coordinates [6]. Presented method can be further extended to include sophisticated universe models with spacetimes defined locally by the Robertson-Walker metrics.

In conclusion, since quantum description of a system is more fundamental than the classical one, compatibility of spacetime with quantization procedure might be used in the search for the topology of the Universe.

We suggest to relate the universe models having topologies compatible with quantum dynamics of a test particle (like V_h case) with the search for the topology based on fluctuations in the cosmic microwave background [7].

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